Period doubling cascade in glow discharges: Local versus global differential conductivity

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Short planar glow discharges coupled to a resistive layer exhibit a wealth of spontaneous spatiotemporal patterns. Due to similarities with other pattern forming systems that are described by reaction-diffusion models, several authors have tried to derive such models from discharge physics. We investigate the temporal oscillations of the discharge system and find a cascade of period doubling events. This shows that the inner structure of the discharge is more complex than can be described by a two-component reaction-diffusion-model with negative differential conductivity. We also derive an alternative reduced model.

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I. PATTERN FORMATION IN SHORT "BARRIER" DISCHARGES

Glow discharges are part of our daily environment in conventional and energy saving lamps, beamers, flat TV screens, car and street lamps, as well as in various industrial applications. While applications typically try to avoid any instabilities, experiments actually exhibit a realm of spontaneous pattern formation, see, e.g., Ref. [1].

While composite gases or complex external electric circuits with ac drive can create many additional structures, even a simple planar discharge gap filled with pure nitrogen and subject to dc drive already can form rich spatiotemporal structures. An interesting series of experiments has been performed on such glow discharges with wide lateral extension [2–13] where the formed patterns were explored very systematically. The observed structures (stripes, hexagons, spirals, spots, homogeneous oscillations, etc.) resemble those observed in Rayleigh-Benard convection in flat cells, in electroconvection in nematic liquid crystals, or in various chemical or biological pattern forming systems. In comparison to the other systems, the glow discharge system has the advantage of particular convenient experimental handling and time scales [14]. In addition to structures familiar from other physical systems, glow discharges continue to exhibit new structures that might be specific for this system [2–13]. We will focus on the experiment in Ref. [9], where a complete phase diagram of different patterns was identified: homogeneous stationary and homogeneous oscillating modes, patterns with spatial and spatiotemporal structures, etc.

As said above, the gas discharge patterns show many similarities with patterns formed in other physical, chemical or biological systems that can be described by reaction-diffusion models. This observation suggests that also to these gas discharges, effective reaction-diffusion models could be applicable. They would describe the phenomena in the two transversal directions of the layered structure. Negative differential conductivity of the glow discharge should then be the driving force of pattern formation. A number of authors [4,15–22] actually have aimed at deriving such models from gas discharge physics, but not in a systematic manner.

II. SUMMARY OF OUR RESULTS

In the present paper, we first examine the concepts of two-component reaction-diffusion models with negative differential conductivity. We then solve a simple explicit model for the gas discharge layer and confront the results with those of reaction-diffusion models. The analysis is performed in a parameter range where the discharge exhibits spontaneous temporal oscillations but no spatial structures transverse to the current [9]. This means that the diffusion part of the model is not relevant, but, of course, our findings generalize to cases with spatial structure.

In short, we find the following.

(i) A discharge on the transition from Townsend to glow discharge can combine a positive local differential conductivity with a negative global differential conductivity.

(ii) A glow discharge in a simple electric circuit shows more complex behavior than can be expected from the proposed reaction-diffusion models[4,15–22] for voltage U and current J with (global) negative differential conductivity $dU/dJ<0$.

(iii) In particular, the system can show a cascade of period doubling bifurcations. Period doubling actually has been observed experimentally in glow discharges, but in more complex geometries and in longer systems [23,24].

(iv) Finally, we derive an effective dynamical model in terms of a parameter and a function by adiabatic elimination of the electrons. There is no systematic way to reduce this model to a simpler one [4,15–22] with two scalar parameters such as voltage U and current J. We draw this conclusion both from direct analysis and from the occurence of period doubling in the numerical solutions.

III. PREVIOUSLY SUGGESTED REDUCTIONS TO REACTION-DIFFUSION MODELS

To be precise, in the experiments of Refs. [2–13], a planar glow discharge layer with short length in the forward direction and wide lateral dimensions is coupled to a semiconductor layer with low conductivity. The whole structure is sandwiched between two planar electrodes to which a dc voltage

U_t is applied. Theoretical predictions on how the different spatiotemporal patterns depend on the parameters of the gas discharge hardly exist. In Refs. [4,16–18], an effective reaction-diffusion model in the two dimensions transversal to the current is proposed. Roughly, it consists of two nonlinear partial differential equations for the current J and the voltage U of the form

$$\partial_t U(x,y,t) = \mathcal{F}(U,J), \quad \partial_t J(x,y,t) = \mathcal{G}(U,J) \tag{1}$$

where the nonlinear operators $\mathcal{F}$ and $\mathcal{G}$ contain spatial derivatives ∂_x, ∂_y and possibly also integral kernels. The model is of reactor-inhibitor form as studied extensively in the context of chemical and biological systems in the past decades. If applicable to gas discharges, this identification lays a connection to a realm of analytical and numerical results on reaction-diffusion systems.

To test whether a model such as Eq. (1) is applicable to the gas discharge system, we will focus on its temporal oscillations that can occur in a spatially completely homogeneous mode [9]; hence a one-dimensional approximation is appropriate. Similar oscillations have been observed in Refs. [19,20,25,26], and similar effective models for current J and voltage U of the general form (1) have been proposed in Refs. [15,19–22].

Why have different authors come up with the same type of model? The equation for U directly results from the simplest form of an external electric circuit: a semiconductor layer of thickness d_s, linear conductivity σ_s, and dielectricity constant ϵ_s will evolve as

$$\partial_t U = \frac{U_t - U - R_s J}{T_s}, \tag{2}$$

where U_t is the voltage on the total system, J is the total current, and $U = \int_0^{d_g} E\, dz$ is the voltage over the gas discharge which is the electric field E integrated in the z direction over the height d_g of the discharge. For the experiments in Ref. [9], $R_s = d_s/\sigma_s$ is the resistance of the whole semiconductor layer, when σ_s is its conductivity, and $T_s = \epsilon_s\epsilon_0/\sigma_s = C_s R_s$ is the Maxwell relaxation time of the semiconductor with dielectricity constant ϵ_s. In other experimental systems, the quantities R_s and T_s can have different realizations. Hence the form of the equation for U in a reaction-diffusion model (1) is clear.

However, the equation for J in a reaction-diffusion model such as Eq. (1) is based on the plausibility of such a model due to analogies with other pattern forming systems and on various attempts to derive such a form with ad hoc assumptions from gas discharge physics. Different choices have been suggested by different authors, but one feature is generic: to be physically meaningful, the current-voltage characteristics of the glow discharge has to be a stationary solution, so $\mathcal{G}(U,J) = 0$ on the characteristics. Beyond that, there are different suggestions for the functional form of $\mathcal{G}$ and the intrinsic time scale.

If a model such as Eq. (1) is applicable to oscillations in glow discharge systems, then the following predictions apply.

(1) An oscillation can only occur in a region of negative differential conductivity $dU/dJ < 0$ of the glow discharge characteristics.

(2) Only a single period can be formed, period doubling is not possible, since this would require at least three independent parameters.

(3) In a phase space plot in U and J, the trajectory of an oscillation can intersect the load line $U = U_t - R_s J$ only parallel to the J axis (since $\partial_t U = 0$ and $\partial_t J \neq 0$), and it can intersect the characteristics of the glow discharge $U = U(J)$ only parallel to the U axis (since $\partial_t U \neq 0$ and $\partial_t J = 0$).

IV. THE FULL DISCHARGE MODEL

We now introduce the simplest classical model for a glow discharge [27–29], solve it numerically, and confront its results with the predictions above. A discharge between Townsend and glow regime consists of a gas with Ohmic conductivity for the rare charged particles, electrostatic space charge effects and two ionization mechanisms, namely, impact ionization by accelerated electrons in the bulk of the discharge (the so-called α process) and secondary emission from the cathode (the γ process). In its simplest form, it can be modeled by continuity equations for electron particle density n_e and ion particle density n_+

$$\partial_t n_e + \boldsymbol{\nabla} \cdot \mathbf{J}_e = S, \quad \partial_t n_+ + \boldsymbol{\nabla} \cdot \mathbf{J}_+ = S, \tag{3}$$

and the Poisson equation for the electric field $\mathbf{E}$ in electrostatic approximation

$$\boldsymbol{\nabla} \cdot \mathbf{E} = \frac{\mathrm{e}}{\varepsilon_0}(n_+ - n_e), \quad \mathbf{E} = -\boldsymbol{\nabla}\Phi. \tag{4}$$

The particle currents are approximated as purely Ohmic:

$$\mathbf{J}_e = -\mu_e n_e \mathbf{E}, \quad \mathbf{J}_+ = \mu_+ n_+ \mathbf{E}. \tag{5}$$

The source of particles in the continuity equation (3) is written as a sum of generation by impact ionization in Townsend approximation and recombination

$$S = |n_e \mu_e E|\ \alpha_0 \mathrm{e}^{-E_0/|E|} - \beta n_e n_+. \tag{6}$$

Finally, the secondary emission from the cathode enters as a boundary condition at the position d_g of the cathode

$$\mu_e n_e(d_g,t) = \gamma \mu_+ n_+(d_g,t). \tag{7}$$

This is the classical glow discharge model [27–29].

We reduce the problem to one spatial dimension z transverse to the layers which is an excellent approximation for the experimentally observed homogeneous oscillations [9]. Furthermore, we introduce dimensionless quantities as in [29] by rescaling all parameters and fields as $z = r_z/X_0$, $\tau = t/t_0$, $L = d_g/X_0$, $\sigma(z,\tau) = n_e(r_z,t)/n_0$, $\rho = n_+/n_0$, $\mathcal{E} = E_z/E_0$ with the scales $X_0 = \alpha_0^{-1}$, $t_0 = (\alpha_0 \mu_e E_0)^{-1}$, and $n_0 = \epsilon_0 \alpha_0 E_0/\mathrm{e}$. A key role is played by the small parameter $\mu = \mu_+/\mu_e$, which is the mobility ratio of ions and electrons.

The gas discharge layer is now modeled by

$$\partial_\tau \sigma = \partial_z(\mathcal{E}\sigma) + \sigma\mathcal{E}\alpha(\mathcal{E}), \quad \alpha(\mathcal{E}) = e^{-1/|\mathcal{E}|}, \tag{8}$$

$$\partial_\tau \rho = -\mu \partial_z(\mathcal{E}\rho) + \sigma\mathcal{E}\alpha(\mathcal{E}), \tag{9}$$

$$\rho(0,\tau) = 0, \tag{10}$$

$$\sigma(L,\tau) = \gamma\mu\rho(L,\tau), \tag{11}$$

$$\rho - \sigma = \partial_z \mathcal{E}, \tag{12}$$

where recombination was neglected [$\beta=0$ in Eq. (6), a discussion of this approximation follows below], while the external circuit is described by

$$\partial_\tau \mathcal{U} = \frac{\mathcal{U}_t - \mathcal{U} - R_s j}{\tau_s}, \quad \mathcal{U}(\tau) = \int_0^L \mathcal{E}(z,\tau)dz \tag{13}$$

with the dimensionless voltage $\mathcal{U}=U/(E_0X_0)$, time scale $\tau_s = T_s/t_0$, and resistance $\mathcal{R}_s=R_s/R_0$, $R_0=X_0/(e\mu_e n_0)$ and with a spatially conserved total current

$$j(\tau) = \partial_\tau \mathcal{E} + \mu\rho\mathcal{E} + \sigma\mathcal{E}, \quad \partial_z j(\tau) = 0, \tag{14}$$

where $\partial_z\, j=0$ follows from Eqs. (8), (9), and (12) as usual.

As a result, the gas discharge is parametrized by the three dimensionless parameters of system length over ionization length L, secondary emission coefficient γ, and mobility ratio μ (as discussed in [29,30]), and the external circuit is parametrized by relative resistance $\mathcal{R}_s$, ratio of time scales τ_s, and dimensionless applied voltage $\mathcal{U}_t$. For calculational purposes, the ion density ρ can be completely eliminated from the one-dimensional gas discharge equations (8), (9), (11), and (12) with the help of the Poisson equation (12) and the total current j, see Ref. [29]. The result are two equations of motion for $\partial_\tau\sigma$ and $\partial_\tau\mathcal{E}$. In our numerical calculations, the system was implemented in this form. Our choice of parameters was guided by the experiments in Ref. [9]: we chose the secondary emission coefficient $\gamma=0.08$, the mobility ratio $\mu=0.0035$ for nitrogen, and the dimensionless system size $L=50$ which amounts to 1.4 mm at a pressure of 40 mbar. The external circuit has $\mathcal{R}_s=30597$, $\tau_s=7435$, and a dimensionless total voltage $\mathcal{U}_t$ in the range between 18 and 20. This corresponds to a GaAs layer with $\epsilon_s=13.1$, conductivity $\sigma_s = (2.6\times10^5\ \Omega\ \mathrm{cm})^{-1}$ and thickness $d_s=1.5$ mm, and a voltage range between 513 and 570 V.

V. SOLUTIONS OF THE DISCHARGE MODEL: A PERIOD DOUBLING CASCADE

Figure 1 shows electric current j and voltage on the gas discharge $\mathcal{U}$ as a function of time for a total stationary voltage $\mathcal{U}_t=19$ applied to the complete system of gas discharge and semiconductor layer. The system exhibits spontaneous oscillations with sharp current peaks: when the voltage $\mathcal{U}$ on the gas layer becomes high enough, the discharge ignites. The conductivity of the gas increases rapidly and produces a current pulse that deposits a surface charge on the gas-semiconductor interface. Therefore the voltage $\mathcal{U}$ over the gas layer breaks down. Due to the low conductivity of the semiconductor, the voltage $\mathcal{U}$ recovers only slowly. Eventually the gas discharge ignites again, and the cycle is repeated.

Note that the oscillations in Fig. 1 are not quite periodic. This is not due to initial transients since the system is observed after the long relaxation time $\tau=4.745\times10^6$. The nature of this temporal structure becomes clear when the trajectory is plotted in the plane spanned by current j and voltage $\mathcal{U}$ in Fig. 2(c). The figure contains the data of the time span from $\tau=3\times10^6$ to 6×10^6 which amounts to approximately 90 current pulses. The phase space plot shows that the system is actually periodic, with a period of eight current pulses. Fig. 1 shows precisely one period.

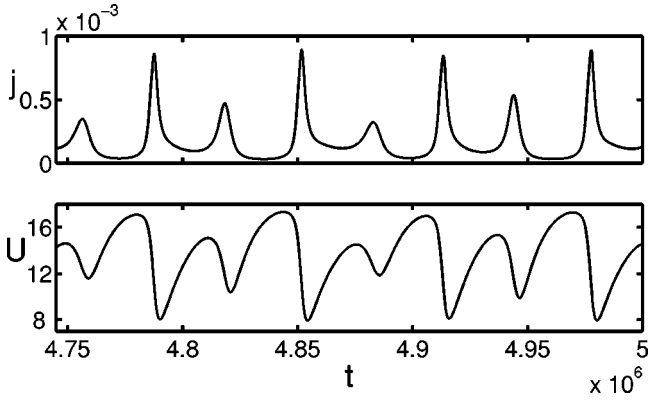

FIG. 1. Spontaneous oscillations of current j and voltage $\mathcal{U}$ as a function of time τ for $\gamma=0.08$, $\mu=0.0035$, $L=50$, $\mathcal{R}_s=30597$, $\tau_s=7435$, and applied total voltage $\mathcal{U}_t=19$.

This discovery raises the question whether our system actually follows the well-known scenario of period doubling. Indeed, it does. Figure 2(a) for $\mathcal{U}_t=18$ shows an oscillation where one current pulse is repeated periodically as observed experimentally in Ref. [9]. For $\mathcal{U}_t=18.5$, a period consists of two current pulses as can be seen in Fig. 2(b). For $\mathcal{U}_t=19$, the period is 8 pulses as in Fig. 1 and 2(c). For $\mathcal{U}_t=20$, the system seems to have reached the chaotic state as can be seen in Fig. 2(d).

A detailed comparison of the experiments in [9] with simple oscillations as in Fig. 2(a) will be given elsewhere [31], and we only state here that there is semiquantitative agreement of several features. Here we emphasize that period doubling events in glow discharges have been observed experimentally in other systems [23,24]. However, this was always in systems with more complicated geometries such as long narrow tubes, and the authors allude to general knowledge on nonlinear dynamics rather than to solutions of explicit models. We state that period doubling can be a generic feature of a simple, strictly one-dimensional glow discharge when coupled to the simple circuit (2). We propose to search experimentally for a period doubling route to chaos in such simple systems which would then allow quantitative comparison with theory.

VI. COMPARISON WITH PREDICTIONS OF REACTION DIFFUSION MODELS

Let us return to the initial question: is a two-component reaction diffusion model such as Eq. (1) with negative differential conductivity appropriate for the present system? At the end of Sec. III, we gave a list of three predictions for the reaction diffusion model (1) to be applicable. Prediction 2 is falsified by the observation of period doubling. Prediction 3

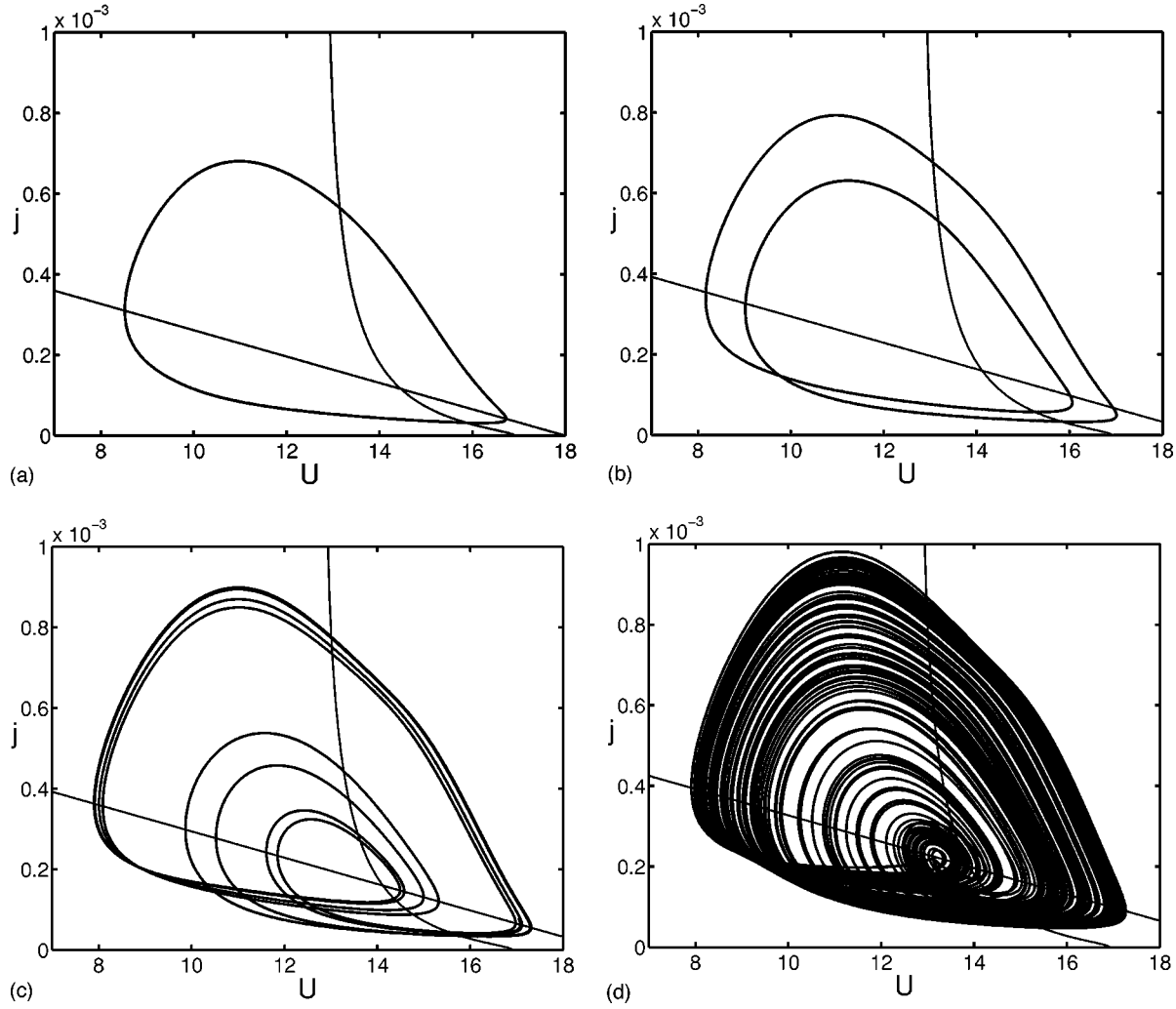

FIG. 2. Phase space plots of the trajectories of the oscillations in the plane of current j and voltage $\mathcal{U}$. The time range is $3\times10^6\leqslant\tau\leqslant6\times10^6$ in all figures. Shown are the orbits, the straight load line $\mathcal{U}=\mathcal{U}_t-\mathcal{R}_s j$ and the curved current voltage characteristics $\mathcal{U}=\mathcal{U}(j)$ of the gas discharge [29]. The intersection of load line and characteristics marks the stationary solution of the system. (c) The data of Fig. 1 with total voltage $\mathcal{U}_t=19$, (a) is for $\mathcal{U}_t=18$, (b) for $\mathcal{U}_t=18.5$, and (d) for $\mathcal{U}_t=20$.

is also falsified by a simple check of either of the three figures in Fig. 2: in the upper part of the figures that represent the rapid current pulses, the trajectories definitely do not intersect with the characteristics with the angle prescribed by Eq. (1).

There rests prediction 1: is negative differential conductivity required for the oscillations to occur? We have not found a numerical counter example where oscillations would occur while the current voltage characteristics of the gas discharge shows a positive differential conductivity, but we have found no reason to exclude its existence. Furthermore we note that the characteristics are a global property of the whole discharge layer with its boundary conditions [29] while the local differential conductivity in our model is always positive: the field dependent stationary ionization is $n_+=|\mu_e E|\alpha_0 e^{-E_0/|E|}/\beta$ according to Eq. (6); hence the local conductivity increases monotonically with the applied field $|E|$. The global negative differential conductivity is due to electrode effects being much stronger than bulk recombination β. In the meantime, we have found a counter example where a state with positive differential conductivity is unstable and develops into a limit cycle. Details will be given elsewhere.

VII. AN ALTERNATIVE SYSTEMATIC MODEL REDUCTION

Finally, we have derived an analytical approximation of the model (8)–(14) that can be confronted with the suggested reaction-diffusion form (1). The reduction is based on an adiabatic elimination of the electrons.

Basically, close to the stationary state, the electric particle current is conserved. Close to the anode $z=0$, this current is mainly carried by the electrons $\sigma\mathcal{E}$ and close to the cathode $z\to L$, it is mainly carried by the ions $\mu\rho\mathcal{E}$. The electric field stays of order unity throughout the system. Therefore $\mu\rho$ should scale similar to σ: $O(\mu\rho)=O(\sigma)$. Now the field $\mathcal{E}$ and the system size L are of order unity, and therefore also $\partial_z\mathcal{E}=\rho-\sigma$. The conclusion is that $\rho=O(1)$, and hence $\sigma=O(\mu)$. Therefore we substitute $s=\sigma/\mu$ where s is now of order

unity. This expresses more clearly that the electrons contribute to the current in order unity, but to the space charge only in order μ. To focus on the dynamics of the ions, a time scale $\overline{\tau}=\mu\tau$ is chosen. Then the model attains the form

$$\mu\ \partial_{\overline{\tau}} s = \partial_z(\mathcal{E}s) + s\mathcal{E}\alpha(\mathcal{E}), \tag{15}$$

$$\partial_{\overline{\tau}}\rho = -\partial_z(\mathcal{E}\rho) + s\mathcal{E}\alpha(\mathcal{E}), \tag{16}$$

$$\rho(0,\overline{\tau}) = 0, \tag{17}$$

$$s(L,\overline{\tau}) = \gamma\rho(L,\overline{\tau}), \tag{18}$$

$$\rho - \mu s = \partial_z\mathcal{E}. \tag{19}$$

This rescaling allows to take two essential approximations in the limit of small μ: First, s evolves on the short time scale $\tau=\overline{\tau}/\mu$ while ρ evolves on the long time scale $\overline{\tau}$. For small μ, the electrons therefore can be eliminated adiabatically from the ion motion. Hence on the ion time scale $\overline{\tau}$ (and as long as $\mathcal{E}\neq 0$), the electron distribution is equilibrated or "slaved" and we can approximate Eq. (15) by

$$[\partial_z + \alpha(\mathcal{E})][s\mathcal{E}] \approx 0. \tag{20}$$

Second, the contribution of the electrons to the space charge is negligible for small μ; hence Eq. (13) can be approximated by

$$\rho \approx \partial_z\mathcal{E}. \tag{21}$$

This can also be understood as follows: electrons and ions are generated in equal amounts, but the electrons rapidly leave the system while the ions move slowly and therefore reach much higher concentrations.

As a consequence of the differential equation (20) and the boundary condition (18), the electron current at position z now can be completely expressed by the instantaneous ion density at the boundary L and the field $\mathcal{E}$ between z and L as

$$[s\mathcal{E}](z) = \gamma\ \rho(L)\mathcal{E}(L)e^{\int_z^L \alpha[\mathcal{E}(z)]dz}. \tag{22}$$

Using Eqs. (16) and (20), the equation of motion for the ions becomes

$$\partial_{\overline{\tau}}\rho = -\partial_z(\rho\mathcal{E} + [s\mathcal{E}]), \quad \rho = \partial_z\mathcal{E} \tag{23}$$

with $[s\mathcal{E}]$ from (22).

Because of Eq. (21), ρ (or alternatively $\mathcal{E}$) can be eliminated from the equations. The result is a rephrasing of the model completely in terms of the field $\mathcal{E}$ and its differentials and integrals

$$\partial_{\overline{\tau}}\mathcal{E} + \partial_z\frac{\mathcal{E}^2}{2} + \gamma\ \partial_z\frac{\mathcal{E}^2}{2}\bigg|_L e^{\int_z^L \alpha[\mathcal{E}(z)]dz} = j(\overline{\tau}), \tag{24}$$

$$\partial_{\overline{\tau}}\mathcal{U}(\overline{\tau}) = \frac{\mathcal{U}_t - \mathcal{U}(\overline{\tau}) - \mathcal{R}_s j(\overline{\tau})}{\mu\tau_s}, \tag{25}$$

where

$$\mathcal{U}(\overline{\tau}) = \int_0^L \mathcal{E}(z,\overline{\tau})dz, \tag{26}$$

and

$$j(\overline{\tau}) = \partial_{\overline{\tau}}\mathcal{E}|_L + (1+\gamma)\ \partial_z\frac{\mathcal{E}^2}{2}\bigg|_L. \tag{27}$$

Let us decompose the field into its value on the cathode

$$\mathcal{E}_L(\overline{\tau}) = \mathcal{E}(L,\overline{\tau}) \tag{28}$$

and the remainder

$$\mathcal{E}(z,\overline{\tau}) = \mathcal{E}_L(\overline{\tau}) + \epsilon(z,\overline{\tau})\ \text{ with } \epsilon|_L = 0, \tag{29}$$

where the ion density can be recovered through $\partial_z\epsilon=\rho$. The equation of motion for ϵ becomes

$$\partial_{\overline{\tau}}\epsilon = \mathcal{E}_L\partial_z\epsilon|_L - \mathcal{E}_L\partial_z\epsilon - \epsilon\partial_z\epsilon + \gamma\mathcal{E}_L\partial_z\epsilon|_L\left(1 - e^{\int_z^L dy\ \alpha(\mathcal{E}_L+\epsilon)}\right). \tag{30}$$

Note that if $\epsilon(L,t)=0$ initially at $t=0$, then the equation of motion for ϵ conserves this property for all times, as required by Eq. (29). For the equation of motion for $\mathcal{E}_L$, first the potential $\mathcal{U}(\overline{\tau})$ has to be expressed in terms of $\mathcal{E}_L$ and ϵ

$$\mathcal{U}(\overline{\tau}) = L\mathcal{E}_L(\overline{\tau}) + \int_0^L dz\ \epsilon(z,\overline{\tau}). \tag{31}$$

Equation (25) for $\partial_{\overline{\tau}}\mathcal{U}$ requires that we calculate

$$\int_0^L dz\ \partial_{\overline{\tau}}\epsilon = \frac{\epsilon^2|_0}{2} + \mathcal{E}_L\epsilon_0 + L\mathcal{E}_L\rho_L \times\left(1 + \gamma - \gamma\int_0^L \frac{dz}{L}e^{\int_z^L dy\ \alpha(\mathcal{E}_L+\epsilon)}\right), \tag{32}$$

where we introduced the short-hand notation

$$\rho_L = \partial_z\epsilon|_L. \tag{33}$$

Introducing the small parameter $c=L\mu\tau_s/\mathcal{R}_s$ which is the ratio of capacitances of semiconductor and gas discharge layers, we find for $\mathcal{E}_L$

$$\partial_{\overline{\tau}}\mathcal{E}_L = -(1+\gamma)\mathcal{E}_L\rho_L + \frac{\mathcal{U}_t - L\mathcal{E}_L - \int_0^L dz\epsilon}{(1+c)\mathcal{R}_s} + \frac{c}{1+c}\left(\mathcal{E}_L\rho_L\gamma\int_0^L \frac{dz}{L}e^{\int_z^L dy\ \alpha} - \frac{\epsilon^2|_0 + 2\mathcal{E}_L\,\epsilon|_0}{2L}\right). \tag{34}$$

The structure of the explicit (but somewhat lengthy) equations (30) and (34) is

$$\partial_{\overline{\tau}}\epsilon(z,\overline{\tau}) = F(\epsilon,\mathcal{E}_L) \quad \text{and} \quad \partial_{\overline{\tau}}\mathcal{E}_L(\overline{\tau}) = G(\epsilon,\mathcal{E}_L). \tag{35}$$

So this is not a system of two scalars such as previous authors have suggested for a reaction diffusion model, but a system of the scalar $\mathcal{E}_L$ and the function ϵ.

The electric field $\mathcal{E}_L$ at the cathode $z=L$ determines the secondary emission γ as well as the local bulk ionization rate

α, and it is closely related to the potential $\mathcal{U}$ over the gas discharge. We therefore take it as the first characteristic scalar variable for the state of the system. A second such scalar variable could be the ion density ρ_L at the cathode, since it characterizes the conductivity and therefore also the electric particle current $\rho_L\mathcal{E}_L + [s\mathcal{E}]|_L = (1+\gamma)\rho_L\mathcal{E}_L$ at the cathode. A two-component reaction-diffusion model could therefore consist of equations for the two scalars $\mathcal{E}_L$ and ρ_L. But can it be derived from the present equations?

Taking the spatial derivative of Eq. (30) at L, we find the equation of motion for ρ_L

$$\partial_{\overline{\tau}}\rho_L = -\rho_L^2 - \mathcal{E}_L\,\partial_z\rho|_L + \gamma\rho_L\mathcal{E}_L\alpha(\mathcal{E}_L). \tag{36}$$

The evolution of ρ_L depends on its local value, on the value of the local field $\mathcal{E}_L$ and also on the local derivative of ρ. If this derivative $\partial_z\rho|_L = \partial_z^2\epsilon|_L$ could be neglected, we would have derived a two-component model.

However, it is not possible to neglect $\partial_z\rho|_L$. This can be easily seen in the Townsend limit of very small space charges where the term $-\rho_L^2$ in (36) can be neglected. Whether ρ_L grows or decays, therefore depends on the sign of $\gamma\rho_L\alpha(\mathcal{E}_L) - \partial_z\rho|_L$, i.e., on the local balance of generation and transport of ions.

An equation of motion for $\partial_z\rho|_L$ in turn would depend on $\partial_z^2\rho|_L$ etc., so an infinite hierarchy of equations would have to be considered. This observation, of course, corresponds to the fact that the state of the gas discharge has been characterized by the full function $\epsilon(z,\overline{\tau})$ or $\rho(z,\overline{\tau})$ throughout the gas gap, where the ion distribution ρ through the gap carries the memory of previous states of the system.

VIII. SUMMARY AND OUTLOOK

We have recalled experimental observations of spontaneous pattern formation in layered gas discharges in the regime between Townsend and glow discharge. Due to similarities with pattern formation in reaction-diffusion models, a number of authors have suggested effective reaction-diffusion models for these discharges. The predictions of such models are summarized and confronted with solutions of a full gas discharge model where we focus on spatially homogeneous temporal oscillations. First, we observe empirically that the predictions of two-component reaction-diffusion models are in variance with the numerical solutions of the full discharge model. Second, we reduce the discharge model by an adiabatic elimination of the electron motion and argue that the resulting mathematical and physical structure is too rich for a reduction to only two relevant variables.

Along the way, we also identify a period doubling cascade in the oscillations of our simple model. Period doubling events have been observed in more complex experimental setups in gas discharges, but it would be most interesting to search for this behavior in the simple experiments of type [9].

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[1] U. Kogelschatz, Plasma Chem. Plasma Process. **23**, 1 (2003).
[2] Yu. A. Astrov *et al.*, Phys. Lett. A **211**, 184 (1996).
[3] Yu. A. Astrov *et al.*, Phys. Rev. Lett. **78**, 3129 (1997).
[4] Yu. A. Astrov and Y. A. Logvin, Phys. Rev. Lett. **79**, 2983 (1997).
[5] E. Ammelt *et al.*, Phys. Rev. E **55**, 6731 (1997).
[6] Y. A. Astrov *et al.*, Phys. Rev. Lett. **80**, 5341 (1998).
[7] E. Ammelt *et al.*, Phys. Rev. E **58**, 7109 (1998).
[8] C. Strümpel *et al.*, Phys. Rev. E **61**, 4899 (2000).
[9] C. Strümpel *et al.*, Phys. Rev. E **62**, 4889 (2000).
[10] C. Strümpel *et al.*, Phys. Rev. E **63**, 026409 (2001).
[11] Y. A. Astrov and H.-G. Purwins, Phys. Lett. A **283**, 349 (2001).
[12] C. Strümpel *et al.*, Phys. Rev. E **65**, 066210 (2002).
[13] E. L. Gurevich *et al.*, Phys. Lett. A **307**, 299 (2003).
[14] W. Breazeal, K. M. Flynn, and E. G. Gwinn, Phys. Rev. E **52**, 1503 (1995).
[15] K. G. Müller, Phys. Rev. A **37**, 4836 (1988).
[16] H.-G. Purwins *et al.*, Phys. Lett. A **136**, 480 (1989).
[17] C. Radehaus *et al.*, Phys. Rev. A **42**, 7426 (1990).
[18] C. Radehaus *et al.*, Phys. Rev. A **45**, 2546 (1992).
[19] A. V. Phelps, Z. L. Petrovic, and B. M. Jelenkovic, Phys. Rev. E **47**, 2825 (1993).
[20] Z. L. Petrovic, I. Stefanovic, S. Vrhovac, and J. Zivkovic, J. Phys. IV **7**, C4, 341 (1997).
[21] V. I. Kolobov and A. Fiala, Phys. Rev. E **50**, 3018 (1994).
[22] R. Sh. Islamov, Phys. Rev. E **64**, 046405 (2001).
[23] T. Braun *et al.*, Phys. Rev. Lett. **59**, 613 (1987).
[24] J. Qin *et al.*, Phys. Rev. Lett. **63**, 163 (1989).
[25] Z. L. Petrovic and A. V. Phelps, Phys. Rev. E **47**, 2806 (1993).
[26] I. Pérès and L. C. Pitchford, J. Appl. Phys. **78**, 774 (1995).
[27] A. von Engel and M. Steenbeck, *Elektrische Gasentladungen* (Springer, Berlin, 1934), Vol. II.
[28] Y. P. Raizer, *Gas Discharge Physics*, 2nd corrected printing (Springer, Berlin, 1997).
[29] D. D. Šijačić and U. Ebert, Phys. Rev. E **66**, 066410 (2002).
[30] Yu. P. Raizer, U. Ebert, and D. D. Šijačić, Phys. Rev. E **70**, 017401 (2004).
[31] D. D. Šijačić, U. Ebert, and I. Rafatov (in preparation).